INTRODUCTION TO ALGEBRAIC NUMBER THEORY 2018 – ENDTERM EXAM

- *Questions are worth a total of* 68 *points. You will be graded out of* 50 *(i.e. a score greater than* 50 *is treated as* 50*).*
- *You are allowed one handwritten A4 sheet (both sides) in the hall.*

Problem 1 (1.5 point each, maximum 12 points)**.** Answer in True/False. If the statement is False, give a counter example (or a proof/explanation). If it is True, no proof is necessary.

(1) All primes of a Dedekind domain are maximal.
(2) Let R be any ring, not necessarily a domain, and $S \subset R$ be a multiplicative set containing 1. Then the natural map of the ring to it's localization $R \to S^{-1}R$ is injective.
(3) The ring $\mathbb{Z}[\sqrt{2}, \sqrt{3}]$ is a Dedekind domain.
(4) Let L/K be an extension of number fields and $\mathcal{O}_K \subset K$ resp. $\mathcal{O}_L \subset L$ corresponding ring of integers. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{O}_K$. Then $\mathfrak{a} \mid \mathfrak{b}$ if and only if $\mathfrak{a}\mathcal{O}_L \mid \mathfrak{b}\mathcal{O}_L$.
(5) Let K be a number field of degree n over $\mathbb{Q}$. If $\mathcal{O}_K$ is the ring of integers in K then the quotient $\mathcal{O}_K/p\mathcal{O}_K$ is a finite ring of cardinality p^n.
(6) Let L be a number field and $\mathfrak{a}$ an ideal of the ring of integers $\mathcal{O}_L \subset L$ and X be the set of integers $\{\Delta(\alpha_1, ..., \alpha_n)\big|\alpha_1, ..., \alpha_n \text{ a basis of } L/\mathbb{Q} \text{ in } \mathfrak{a}\}$ where Δ denotes the discriminant over $\mathbb{Z}$. Then X contains only one of the generators of the discriminant ideal $\Delta(\mathfrak{a}) \subset \mathbb{Z}$.
(7) There is an integer x such that $x^2 \equiv 29 \pmod{31}$.
(8) There is an extension L/K of number fields, such that no prime $\mathfrak{p}$ of the number ring $\mathcal{O}_K$ ramifies in L.
(9) Let $K = \mathbb{Q}[\sqrt{-7}]$ and $\mathcal{O}_K \subset K$ denote the corresponding ring of integers. Then, there are infinitely many units in the ring $\mathcal{O}_K$.
(10) Let K be a ring of integers, $\mathcal{O}_K \subset K$ the ring of integers and $\mathcal{O}_K^* \subset \mathcal{O}_K$ the units in $\mathcal{O}_K$. Let $j : \mathcal{O}_K^* \longrightarrow \prod_{\tau \in G} \mathbb{R}_\tau =: V$ denote the map $x \mapsto (\log|\tau x|)_\tau$ where the coordinate τ varies over $G = \hom(K, \mathbb{C})$. Then $j(\mathcal{O}_K^*)$ is a full lattice in the vector space V.

Problem 2 (10 points)**.** Let I and J be two nonzero fractional ideals of a Dedekind domain R. Let C_R denote the class group of R (that is the group of invertible fractional ideals modulo principal ideals). Show that $I \cong J$ as R modules if and only if I and J represent the same member of C_R.

Problem 3 (10 points)**.** Let p be an odd prime, show that the cyclotomic field of p-th roots of unity contains a unique quadratic extension of $\mathbb{Q}$. Compute the quadratic extension in terms of p.

Problem 4 (12 points)**.** Let $\theta = \sqrt[3]{2}$. Show that the ring of integers $\mathcal{O}$ of the field $\mathbb{Q}(\theta)$ is $\mathbb{Z}[\theta]$ as follows:

- Using the discriminant, find an integer m such that $m\mathcal{O} \subset \mathbb{Z}[\theta] \subset \mathcal{O}$.
- By a direct computation or otherwise show that if $\frac{1}{m}(a + b\theta + c\theta^2)$ is in $\mathcal{O}$, then $m \mid a$, $m \mid b$, $m \mid c$.

Problem 5 (12 points)**.** Let $K = \mathbb{Q}(\sqrt{-97})$. Then is there an ideal $\mathfrak{a}$ in the ring of integers $\mathcal{O}_K \subset K$ such that $N_{K/\mathbb{Q}}(\mathfrak{a}) = (2018)$?

Problem 6 (12 points)**.** Compute class group of $\mathbb{Q}[\sqrt{-30}]$ as follows:

- Use Minkowski bound to show that primes $\mathfrak{p}$ that lie over (2), (3) and (5) generate the class group and calculate the decomposition of (2), (3) and (5) as a product of primes in $\mathbb{Q}[\sqrt{-30}]$.
- Use $N(\sqrt{-30}) = 30 = 2 \cdot 3 \cdot 5$ to compute the decomposition of the principal ideal $(\sqrt{-30})$ as $p_1p_2p_3$ where p_1, p_2, p_3 are distinct prime ideals.
- Show that p_1, p_2, p_3 are not principal and use this to compute the class group.